

THE FOUNDATIONS OF P-CONVEXITY AND P-PURISUBHARMONICTY IN RIEMANNIAN GEOMETRY

F. Reese Harvey and H. Blaine Lawson, Jr.*

ABSTRACT

In this paper we systematically study the notions of a p -plurisubharmonic function and p -convexity in riemannian geometry. New facts of a basic nature are established. In addition, local p -convexity is shown to imply p -convexity (analogous to the Levi problem in complex analysis). A level- p version of the Monge-Ampère operator is introduced. The solution to the Dirichlet problem for all branches of the homogeneous level- p Monge-Ampère equation is given.

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1. Introduction.

On any riemannian n -manifold there are intrinsic notions of p -plurisubharmonicity and p -convexity for integers p between 1 and n . They interpolate between convexity ($p = 1$) and subharmonicity ($p = n$) with $p = n - 1$ an important case. They arise naturally in many situations. One object of this paper is to give a systematic account of the geometry and analysis associated to the ideas in this domain. In part we will gather certain general results of the authors, and others, which have particularly nice applications here. We shall also establish some new results.

The central algebraic idea is that of p -positivity for a quadratic form Q on a finite-dimensional inner product space V . By definition Q is p -positive if the trace of its restriction to every p -dimensional subspace $W \subset V$ satisfies $\text{tr} \{Q|_W\} \geq 0$. This is equivalent to the condition that $\lambda_1 + \cdots + \lambda_p \geq 0$ where $\lambda_1 \leq \cdots \leq \lambda_n$ are the ordered eigenvalues of Q . On any riemannian manifold X , a function $u \in C^2(X)$ is p -plurisubharmonic if its hessian is p -positive. An oriented hypersurface in X is p -convex if its second fundamental form is p -positive. The Riemann curvature R of X is p -positive if for each tangent vector v , the quadratic form $\langle R_{v,\cdot}, v \rangle$ is p -positive.

The smooth p -plurisubharmonic functions are “pluri”-subharmonic in the following sense.

Theorem 2.10. *A function $u \in C^2(X)$ is p -plurisubharmonic if and only if its restriction to every p -dimensional minimal submanifold is subharmonic in the induced metric.*

The notion of p -plurisubharmonicity can be generalized to arbitrary upper semi-continuous $[-\infty, \infty)$ -valued functions using standard viscosity test functions (cf. [CIL], [C]). For $p = 1, n$ this recaptures the classical notions of general convex and subharmonic functions on a riemannian manifold X . The family of p -plurisubharmonic functions, denoted $\text{PSH}_p(X)$, has many of the useful properties of subharmonic functions (see Theorem 7.2). Moreover, the Restriction Theorem 2.10 has a non-trivial extension to general, upper semi-continuous p -plurisubharmonic functions.

Theorem 7.3. *A function $u \in \text{PSH}_p(X)$ if and only if its restriction to every p -dimensional minimal submanifold is subharmonic in the induced metric.*

For each p we introduce a p^{th} Monge-Ampère operator $\text{MA}_p(u)$ given by an explicit polynomial in $\text{Hess } u$. These operators interpolate between the Laplace-Beltrami operator $\Delta_X u = \text{tr}\{\text{Hess } u\}$, for $p = n$, and the standard Monge-Ampère operator $\det\{\text{Hess } u\}$ for $p = 1$. One is led naturally to the p^{th} Monge-Ampère problem: To find p -plurisubharmonic functions u which satisfy $\text{MA}_p(u) = 0$. This constitutes the main branch of the problem. There is a hierarchy of problems corresponding to other branches of the locus $\mathfrak{M}_p(A) = 0$ where $\text{MA}_p(u) = \mathfrak{M}_p(\text{Hess } u)$ (see Section 3). The solution of the associated Dirichlet Problem in all cases will be discussed at the end of the introduction and in Section 9.

The smooth p -plurisubharmonic functions can be used to introduce a notion of p -convexity as follows. Given a compact subset $K \subset X$, define the p -convex hull of K to be the set \widehat{K} of points $x \in X$ such that

$$u(x) \leq \sup_K u$$

for all smooth p -plurisubharmonic functions u on X . Then X is said to be p -convex if

$$K \subset\subset X \quad \Rightarrow \quad \widehat{K} \subset\subset X.$$

We have the following result.

Theorem 4.4. *A riemannian manifold X is p -convex if and only if X admits a smooth p -plurisubharmonic proper exhaustion function.*

A domain $\Omega \subset X$ is said to be *locally p -convex* if each point $x \in \partial\Omega$ has a neighborhood U such that $\Omega \cap U$ is p -convex. Note that p -convex domains are locally p -convex (see (4.1)). The following converse is an analogue of Levi Problem in complex analysis, and is one of the new results of this paper.

Theorem 4.7. *Let $\Omega \subset\subset \mathbf{R}^n$ be a domain with smooth boundary. If Ω is locally p -convex, then Ω is p -convex.*

There is also a notion of p -convexity for the boundary. Let II denote the second fundamental form of the boundary $\partial\Omega$ with respect to the interior normal. Then the boundary $\partial\Omega$ is p -convex if II_x is p -positive at each point $x \in \partial\Omega$.

Theorem 4.9. *Let $\Omega \subset\subset \mathbf{R}^n$ be a domain with smooth boundary. If Ω is locally p -convex, then $\partial\Omega$ is p -convex.*

From Theorem 4.10 one then concludes that for such domains Ω ,

$$\Omega \text{ is } p\text{-convex} \quad \Longleftrightarrow \quad \Omega \text{ is locally } p\text{-convex} \quad \Longleftrightarrow \quad \partial\Omega \text{ is } p\text{-convex}.$$

A quadratic form A on an inner product space V is said to be **strictly** p -positive if $\text{tr} \{A|_W\} > 0$ for all p -planes $W \subset V$. This gives notions of strict p -plurisubharmonicity, strict p -convexity, etc. In Section 5 a number of results concerning strictly p -convex domains and strictly p -convex boundaries are given. A key concept here is that of the *core* of X , a subset which governs the existence of strictly p -plurisubharmonic functions and proper exhaustions (see Theorem 5.4.) The core contains all compact minimal submanifolds without boundary in X . When X is p -convex and the core is empty, X admits a strictly p -plurisubharmonic proper exhaustion function, and standard Morse Theory implies that X has the homotopy-type of a complex of dimension $\leq p - 1$ (cf. [S₁], [Wu]). Strict p -convexity of the boundary $\partial\Omega$ implies that the core is compact and there exists a p -plurisubharmonic proper exhaustion of X which is strict outside a compact set. When the boundary is only p -convex, similar conclusions hold if the riemannian curvature is strictly p -positive near $\partial\Omega$. This result of H. Wu [Wu] is given in Theorem 5.9.

In Section 6 the relationship of p -plurisubharmonic functions to minimal submanifolds and currents is explored. A p -dimensional rectifiable current $T \in \mathcal{R}_p(X)$ on X is called *minimal* if the first variation of the mass of T is zero with respect to deformations supported away from its boundary ∂T (see Definition 6.1).

Corollary 6.5. *Suppose $T \in \mathcal{R}_p(X)$ is a minimal current, and let u be any smooth p -plurisubharmonic function which vanishes on a neighborhood of $\text{supp}(\partial T)$. Then*

$$\text{tr}_{\overrightarrow{T}} (\text{Hess } u) \equiv 0 \quad \text{on } \text{supp}(T).$$

If $T = [M]$ is the current associated to a connected p -dimensional minimal submanifold, and if the p -plurisubharmonic function u and its gradient both vanish at points of ∂M , then

$$u|_M \equiv 0 \quad \text{or, if } \partial M = \emptyset, u|_M \equiv \text{constant}.$$

Corollary 6.6. *Let $K \subset X$ be a compact subset and suppose $T \in \mathcal{R}_p(X)$ is a minimal current such that $\text{supp}(\partial T) \subset K$. Then*

$$\text{supp}(T) \subset \hat{K} \cup \text{Core}(X).$$

This leads to the notion of a *minimal surface hull* of a compact set $K \subset X$, namely the union of the supports of all minimal currents $T \in \mathcal{R}_p(X)$ whose boundaries are supported in K . By Corollary 6.6, this hull is contained in $\hat{K} \cup \text{Core}(X)$.

Much of this discussion carries over to minimal (non-rectifiable) p -currents.

In Section 8 viscosity solutions of the various branches of the homogeneous p^{th} Monge-Ampère equation are discussed.

In Section 9 the solution to the Dirichlet problem for this equation is presented.

Theorem 9.1. *Let $\Omega \subset\subset X$ be a strictly p -convex domain with smooth boundary $\partial\Omega$ in an n -dimensional riemannian manifold X . Fix an integer k with $1 \leq k \leq \binom{n}{p}$. Then for each continuous function $\varphi \in C(\partial\Omega)$, there exists a unique function $u \in C(\bar{\Omega})$ such that:*

- (a) $u|_{\Omega}$ is a solution of the k^{th} branch of the homogeneous p^{th} Monge-Ampère equation,
- (b) $u|_{\partial\Omega} = \varphi$.

Solutions of certain inhomogeneous equations also exist.

In Appendix A the extreme points in the cone $\mathcal{P}_p(V)$ of p -positive quadratic forms in $\text{Sym}^2(V)$ are classified.

Finally we note that the basic notions of p -plurisubharmonicity and p -convexity make sense with the grassmann bundle $G(p, TX)$ replaced by a closed subset $\mathbf{G} \subset G(p, TX)$. There are surprisingly many results which hold in the general context of a “ \mathbf{G} -geometry”. They are discussed in a separate but companion paper [HL₈].

2. Plurisubharmonicity.

Euclidean Space.

Suppose V is an n -dimensional real inner product space, and fix an integer p , with $1 \leq p \leq n$. Let $\text{Sym}^2(V)$ denote the space of symmetric endomorphisms of V . Using the inner product, this space is identified with the space of quadratic forms on V . The notion of p -plurisubharmonicity for a smooth function u on V is defined by requiring that its hessian (i.e., second derivative) belong to a certain subset $\mathcal{P}_p(V) \subset \text{Sym}^2(V)$. To better understand this subset we offer several (equivalent) definitions.

Definition 2.1. Suppose $A \in \text{Sym}^2(V)$. Then $A \in \mathcal{P}_p(V)$, or A is **p -positive**, if the following equivalent conditions hold.

$$(1) \quad \text{tr}_W A \geq 0 \quad \text{for all } W \in G(p, V)$$

$$(2) \quad \lambda_1(A) + \cdots + \lambda_p(A) \geq 0$$

$$(3) \quad D_A \geq 0$$

where:

(1) $G(p, V)$ denotes the set of p -dimensional subspaces of V , and for $W \in G(p, V)$, the W -**trace** of A , denoted $\text{tr}_W A$, is the trace of the restriction $A|_W$ of A to W ,

(2) $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ are the ordered eigenvalues of A , so Condition (2) says that the sum of the p smallest eigenvalues is ≥ 0 ,

(3) $D_A : \Lambda^p V \rightarrow \Lambda^p V$ is the linear action of A as a derivation on the space $\Lambda^p V$ of p -vectors, i.e., on simple p -vectors one has

$$D_A(v_1 \wedge \cdots \wedge v_p) = (Av_1) \wedge v_2 \wedge \cdots \wedge v_p + v_1 \wedge (Av_2) \wedge \cdots \wedge v_p + \cdots + v_1 \wedge v_2 \wedge \cdots \wedge (Av_p).$$

The inner product on V induces an inner product on $\Lambda^p V$, and we have $D_A \in \text{Sym}^2(\Lambda^p V)$, so the notions of non-negativity, $D_A \geq 0$, and positive definiteness, $D_A > 0$, make sense for D_A .

The proof that condition (1), (2) and (3) are equivalent will be given below.

Definition 2.2. (p -plurisubharmonicity). A smooth function u defined on an open subset $X \subset \mathbf{R}^n$ is said to be p -**plurisubharmonic** if $D_x^2 u = \text{Hess}_x u \in \mathcal{P}_p(\mathbf{R}^n)$ for each point $x \in X$.

The next result justifies the terminology.

Proposition 2.3. *A function $u \in C^\infty(X)$ is p -plurisubharmonic if and only if the restriction $u|_{W \cap X}$ is subharmonic for all affine p -planes $W \subset \mathbf{R}^n$. (Here “subharmonic” means that $\Delta_W(u|_{W \cap X}) \geq 0$ where Δ_W is the euclidean Laplacian on the affine subspace W).*

Proof. This is obvious from Condition (2) since with $v = u|_{W \cap X}$, we have $\text{tr}_W D^2 u = \Delta_W v$ on $W \cap X$. ■

Remark 2.4. The endpoint cases are classical.

($p = 1$) Convex Functions. Note that $A \in \mathcal{P}_1 \iff \lambda_{\min}(A) \geq 0 \iff A \geq 0$, so that u is 1-plurisubharmonic $\iff u$ is convex.

($p = n$) Classical Subharmonic Functions. Note that $A \in \mathcal{P}_n \iff \text{tr} A \geq 0$, so that u is n -plurisubharmonic $\iff \Delta u \geq 0$, i.e., u is classically subharmonic.

Consequently, the simplest new case is when $p = 2$ in \mathbf{R}^3 where u is 2-plurisubharmonic \iff the restriction of u to each affine plane in \mathbf{R}^3 is classically subharmonic. One generalization of this case has an interesting characterization.

($p = n - 1$) If $p = n - 1$, then $*$: $\Lambda^1 V \rightarrow \Lambda^{n-1} V$ is an isomorphism. This induces an isomorphism $\text{Sym}^2(\Lambda^{n-1} V) \rightarrow \text{Sym}^2(\Lambda^1 V)$ sending $D_A \mapsto (\text{tr} A)I - A$. Therefore $u \in C^\infty(X)$ is $n - 1$ -plurisubharmonic if and only if

$$(\Delta u)I - \text{Hess } u \geq 0.$$

Note.

(a) It is obvious from Condition (2) that $\mathcal{P}_p(V) \subset \mathcal{P}_{p+1}(V)$, or equivalently, if u is p -plurisubharmonic, then u is $(p+1)$ -plurisubharmonic. In particular, each p -plurisubharmonic function is classically subharmonic, and every convex function is p -plurisubharmonic for all p .

(b) The set $\mathcal{P}_p(V)$ is a closed convex cone with vertex at the origin.

The proof of the equivalence of Conditions (1), (2) and (3) in Definition 2.1 requires some elementary facts. Note that each p -plane $W \subset V$ determines a line $L(W) \subset \Lambda^p V$, namely the line through $v_1 \wedge \cdots \wedge v_p$ where v_1, \dots, v_p is any basis for W . If e_1, \dots, e_n is an orthonormal basis of V , we set $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$ for $I = (i_1, \dots, i_p)$ with $i_1 < i_2 < \cdots < i_p$.

Lemma 2.5. *Given $A \in \text{Sym}^2(V)$, consider $D_A \in \text{Sym}^2(\Lambda^p V)$. Then we have:*

$$(a) \text{ For all } W \in G(p, V), \quad \text{tr}_W A = \text{tr}_{L(W)} D_A. \quad (2.1)$$

(b) If A has eigenvectors e_1, \dots, e_n with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$, then D_A has eigenvectors e_I with corresponding eigenvalues

$$\lambda_I = \lambda_{i_1} + \cdots + \lambda_{i_p}. \quad (2.2)$$

Proof. For (a), note that, if e_1, \dots, e_p is an orthonormal basis of W , then $\text{tr}_{L(W)} D_A = \langle D_A(e_1 \wedge \cdots \wedge e_p), e_1 \wedge \cdots \wedge e_p \rangle = \sum_{j=1}^n \langle e_1 \wedge \cdots \wedge A e_j \wedge \cdots \wedge e_p, e_1 \wedge \cdots \wedge e_p \rangle = \sum_{j=1}^n \langle A e_j, e_j \rangle = \text{tr}_W A$. For (b), compute $D_A e_I = \lambda_I e_I$. ■

Corollary 2.6. *Suppose $A \in \text{Sym}^2(V)$ has ordered eigenvalues $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$. Then*

$$\inf_{W \in G(p, W)} \text{tr}_W A = \lambda_1(A) + \cdots + \lambda_p(A) = \lambda_{\min}(D_A), \quad (2.3)$$

the smallest eigenvalue of D_A .

Proof. Since D_A has eigenvalues λ_I by part (b), the smallest is $\lambda_1(A) + \cdots + \lambda_p(A) = \text{tr}_{L(\overline{W})} D_A$ where $\overline{W} = \text{span}\{e_1, \dots, e_p\}$. Now the smallest eigenvalue of D_A equals the infimum of $\text{tr}_L D_A$ over all lines in $\Lambda_p V$, so in this case it is also the infimum over the restricted set of lines of the form $L(W)$ with $W \in G(p, V)$. By part (a), this proves (2.3). ■

The equivalence of Conditions (1), (2) and (3) in Definition 2.1 is immediate from Corollary 2.6.

Definition 2.7. (p -Harmonic). A smooth function u defined on an open subset $X \subset \mathbf{R}^n$ is p -harmonic if $D_x^2 u \in \partial \mathcal{P}_p$ for all $x \in X$, or equivalently if $\lambda_{\min}(D_{H_x}) = 0$, where $H_x = D_x^2 u$, for all $x \in X$.

Example 2.8. (Radial Harmonics).

(**p = 1**) The function $|x|$ is 1-harmonic on $\mathbf{R}^n - \{0\}$.

(**p = 2**) The function $\log|x|$ is 2-harmonic on $\mathbf{R}^n - \{0\}$.

(**3 ≤ p ≤ n**) The function $-\frac{1}{|x|^{p-2}}$ is p -harmonic on $\mathbf{R}^n - \{0\}$.

Proof. If $u(x) = |x|$, then

$$\text{Hess } u = \frac{1}{|x|} \left(I - \frac{x}{|x|} \circ \frac{x}{|x|} \right). \quad (2.4)$$

If $u(x) = \log|x|$, then

$$\text{Hess } u = \frac{2}{|x|^2} \left(\frac{1}{2} I - \frac{x}{|x|} \circ \frac{x}{|x|} \right). \quad (2.5)$$

If $u(x) = -1/|x|^{p-2}$, then

$$\text{Hess } u = \frac{(p-2)p}{|x|^p} \left(\frac{1}{p} I - \frac{x}{|x|} \circ \frac{x}{|x|} \right). \quad (2.6)$$

Note from (2.4) - (2.6) that the hessian of u is a scalar multiple of $H \equiv \frac{1}{p} \cdot I - e \circ e$ where $e = \frac{x}{|x|}$. The eigenvalues of H are $\frac{1}{p} - 1 = -\frac{p-1}{p}$ and $\frac{1}{p}$ occurring with multiplicity $n-1$. This implies that the eigenvalues of D_H are 0 and 1, and in particular, $\lambda_{\min}(D_H) = 0$. ■

Remark 2.9. For each of the radial functions u in Example 2.8, if $2 \leq p \leq n-1$, the hessian $\text{Hess}_x u$ generates an extremal ray in the cone \mathcal{P}^+ (See Appendix A).

Riemannian Manifolds.

Suppose X is an n -dimensional riemannian manifold. Then the euclidean notions above carry over with $V = T_x X$ and the ordinary hessian of a smooth function replaced by the **riemannian hessian**. For $u \in C^2(X)$ this is a well defined section of the bundle $\text{Sym}^2(TX)$ given on tangent vector fields V, W by

$$(\text{Hess } u)(V, W) = VWu - (\nabla_V W)u, \quad (2.7)$$

where ∇ denotes the Levi-Civita connection. Acting as a derivation, it determines a well defined section $D_{\text{Hess } u}$ of $\text{Sym}^2(\Lambda^p TX)$ for each p , $1 \leq p \leq n$.

Definition 2.2' (p -plurisubharmonicity). A smooth function u on X is said to be **p -plurisubharmonic** if $\text{Hess}_x u$ is p -positive at each point $x \in X$ (see Definition 2.1).

The appropriate geometric objects for restriction are the p -dimensional minimal (stationary) submanifolds of X . In the euclidean case this enlarges the family of affine p -planes used in Proposition 2.3 when $1 < p < n$.

THEOREM 2.10. *A function $u \in C^2(X)$ is p -plurisubharmonic if and only if the restriction of u to every p -dimensional minimal submanifold is subharmonic.*

Proof. Suppose $M \subset X$ is any p -dimensional submanifold, and let H_M denote its mean curvature vector field. Then

$$\Delta_M (u|_M) = \text{tr}_{TM} \text{Hess } u - H_M u. \quad (2.8)$$

In particular, if M is minimal, then

$$\Delta_M(u|_M) = \text{tr}_{TM} \text{Hess } u. \quad (2.9)$$

It is an elementary fact that for every point $x \in X$ and every p -plane $W \subset T_x X$, there exists a minimal submanifold M with $T_x M = \xi$. This is enough to conclude Theorem 2.10 from (2.9). \blacksquare

3. The p^{th} Monge-Ampère Operator.

Given $A \in \text{Sym}^2(V)$, the determinant of $D_A \in \text{Sym}^2(\Lambda^p V)$ defines a homogeneous polynomial of degree $\binom{n}{p}$ on $\text{Sym}^2(V)$.

Definition 3.1. The p^{th} **Monge-Ampère polynomial** on the vector space $\text{Sym}^2(V)$ is defined by

$$\text{MA}_p(A) = \det D_A = \prod_{i_1 < \dots < i_p} \lambda_I(A) \quad (3.1)$$

(where $\lambda_I(A) = \lambda_{i_1}(A) + \dots + \lambda_{i_p}(A)$ as before). The corresponding operator on smooth functions u , namely

$$\text{MA}_p(u) = \det D_{\text{Hess } u} = \prod_{i_1 < \dots < i_p} \lambda_I(\text{Hess } u) \quad (3.2)$$

will be referred to as the p^{th} **Monge-Ampère operator**.

Examples.

- $(p = 1)$ $\text{MA}_1(u) = \det(\text{Hess } u)$ is the standard Monge-Ampère operator.
- $(p = n)$ $\text{MA}_n(u) = \Delta u$ is the Laplacian.
- $(p = n - 1)$ $\text{MA}_{n-1}(u) = \det((\Delta u) \cdot I - \text{Hess } u)$

Definition 3.2. A smooth function u is said to satisfy the **homogeneous p^{th} Monge-Ampère equation** if

$$u \text{ is } p\text{-plurisubharmonic} \quad \text{and} \quad \text{MA}_p(u) = 0.$$

Proposition 3.3. *A smooth function satisfies the homogeneous p^{th} Monge-Ampère equation if and only if it is p -harmonic (see Definition 2.7).*

Note 3.4. In particular each of the radial harmonics given in Example 2.8 is a solution of the homogeneous p^{th} Monge-Ampère equation.

Proof. $H \equiv \text{Hess}_x u \in \partial \mathcal{P}_p \iff H \in \mathcal{P}_p$ and $\text{MA}_p(H) = 0$, since

$$\text{Int } \mathcal{P}_p(V) = \{A \in \text{Sym}^2(V) : \lambda_I(A) > 0 \text{ for all } |I| = p\} \quad (3.3)$$

These definitions clearly carry over to any riemannian manifold X , and we have the following.

Proposition 3.5. *Suppose $u \in C^\infty(X)$ is strictly p -plurisubharmonic. Then the linearization of MA_p at u is elliptic.*

Proof. Fix $a \in X$ and geodesic normal coordinates (x_1, \dots, x_n) at a with $x(a) = 0$. We may assume the coordinate axes are eigendirections of $\text{Hess}_0 u$ so that at 0

$$\text{Hess } u = \sum_{j=1}^n \lambda_j e_j \circ e_j$$

where by assumption $\lambda_j > 0$ for all j . It follows that

$$D_{\text{Hess } u} = \sum_I \lambda_I e_I \circ e_I$$

at 0, with notation as above. One now computes that

$$\left. \frac{d}{dt} \left\{ \det (D_{\text{Hess}(u+tv)}) \right\} \right|_{t=0} = \sum_I \frac{1}{\lambda_I} \text{tr}_{\xi_I} (\text{Hess } v)$$

where $\xi_I = \text{span } e_I$ is the coordinate I -plane. In particular if $\eta = \sum_j \eta_j dx_j$ is a cotangent vector at 0, then the principal symbol of the linearization $L_u(\text{MA}_p)$ of MA_p at u is

$$\sigma_\eta (L_u(\text{MA}_p)) = \sum_I \frac{1}{\lambda_I} |\eta_I|^2$$

where $\eta_I \equiv \sum_{j \in I} \lambda_j e_j$. Evidently $\sigma_\eta (L_f(\text{MA}_p)) = 0$ iff $\eta = 0$. ■

Much can be said about solving the p^{th} Monge-Ampère equation as well as a larger family of equations which we now describe.

Other Branches of the Homogeneous p^{th} Monge-Ampère Equation. Fix $1 \leq p \leq n$. For each integer k , with $1 \leq k \leq \binom{n}{p}$, let $\mathcal{P}_p^{(k)}(V) \subset \text{Sym}^2(V)$ denote the subset of $A \in \text{Sym}^2(V)$ such that the k^{th} smallest p -fold sum of its eigenvalues is ≥ 0 . That is, if the ordered eigenvalues of D_A are $\lambda_{I_1}(A) \leq \dots \leq \lambda_{I_N}(A)$, where $N = \binom{n}{p}$, then

$$\mathcal{P}_p^{(k)}(V) = \{A \in \text{Sym}^2(V) : \lambda_{I_k}(A) \geq 0\}.$$

Note that $\mathcal{P}_p^{(1)}(V) \subset \mathcal{P}_p^{(2)}(V) \subset \dots \subset \mathcal{P}_p^{(n)}(V)$ and $\mathcal{P}_p^{(1)}(V) = \mathcal{P}_p(V)$. The remaining sets $\mathcal{P}_p^{(k)}(V)$, $k > 1$ are not convex. However, the important fact here is that

$$\mathcal{P}_p^{(k)}(V) + \mathcal{P}_p(V) \subset \mathcal{P}_p^{(k)}(V),$$

which follows easily from Definition 2.1(3) since $A \geq 0 \Rightarrow D_A \geq 0$. This fact implies the positivity condition needed for potential theory and the Dirichlet problem, namely

$$\mathcal{P}_p^{(k)}(V) + \mathcal{P}_1(V) \subset \mathcal{P}_p^{(k)}(V).$$

The zero set of MA_p is just the union of the boundaries

$$\{\text{MA}_p(A) = 0\} = \bigcup_{1 \leq k \leq \binom{n}{p}} \partial \mathcal{P}_p(k) \quad (3.4)$$

Definition 3.6. A C^2 -function on an open set $X \subset \mathbf{R}^n$ is said to satisfy the k^{th} **branch of the p^{th} Monge-Ampère equation** if

$$\text{Hess}_x u \in \partial \mathcal{P}_p^{(k)} \text{ for all } x \in X.$$

Our solution to the Dirichlet problem for this equation will be discussed in Section 9.

4. Convexity, Boundary Convexity, and Local Convexity Riemannian Manifolds

Let $\text{PSH}_p^\infty(X)$ denote the smooth p -plurisubharmonic functions on a riemannian manifold X .

Definition 4.1. Given a compact subset $K \subset X$, the **p -convex hull of K** is the set

$$\widehat{K} \equiv \{x \in X : u(x) \leq \sup_K u \text{ for all } u \in \text{PSH}_p^\infty(X)\}$$

Proposition 4.2. *If $M \subset X$ is a compact connected p -dimensional minimal submanifold with boundary $\partial M \neq \emptyset$, then*

$$M \subset \widehat{\partial M}.$$

Proof. Apply Theorem 2.10 and the maximum principle for subharmonic functions on M .
■

Definition 4.3. We say that X is **p -convex** if for all compact sets $K \subset X$, the hull \widehat{K} is also compact.

THEOREM 4.4. *Suppose X is a riemannian manifold. Then:*

- (1) X is p -convex \iff
- (2) X admits a smooth p -plurisubharmonic proper exhaustion function.

Proof. See Theorem 4.4 in [HL₈] for the proof. It is exactly the same proof as the one given for Theorem 4.3 in [HL₁].
■

Condition (2) can be weakened to a local condition at ∞ in the one-point compactification $\overline{X} = X \cup \{\infty\}$. This follows from the next lemma.

Lemma 4.5. *Suppose that $X - K$ admits a smooth p -plurisubharmonic function v with $\lim_{x \rightarrow \infty} v(x) = \infty$ where K is compact. Then X admits a smooth p -plurisubharmonic proper exhaustion function which agrees with v near ∞ .*

Proof. This is a special case of Lemma 4.6 in [HL₈]. ■

Euclidean Space.

We now show that the p -convexity of a compact domain with smooth boundary in euclidean space is a local condition on the domain near the boundary. This result is to some degree analogous to the Levi Problem in complex analysis.

Definition 4.6. A domain $\Omega \subset \mathbf{R}^n$ is **locally p -convex** if each point $x \in \partial\Omega$ has a neighborhood U in \mathbf{R}^n such that $\Omega \cap U$ is p -convex.

Each ball in \mathbf{R}^n is p -convex, and the intersection of two p -convex domains is again p -convex. Therefore

$$\text{If } \Omega \text{ is } p\text{-convex, then } \Omega \text{ is locally } p\text{-convex.} \quad (4.1)$$

Our main result is the converse.

THEOREM 4.7. *Suppose that Ω is a compact domain with smooth boundary. If Ω is locally p -convex, then Ω is p -convex.*

Intermediate between local and global convexity is the notion of boundary convexity. Suppose now that $\partial\Omega$ is smooth.

Denote by $II = II_{\partial\Omega}$ the second fundamental form of the boundary with respect to the **inward pointing** normal n . This is a symmetric bilinear form on each tangent space $T_x\partial\Omega$ defined by

$$II_{\partial\Omega}(v, w) = -\langle \nabla_v n, w \rangle = \langle n, \nabla_v W \rangle$$

where W is any vector field tangent to $\partial\Omega$ with $W_x = w$.

Definition 4.8. The boundary $\partial\Omega$ is **p -convex at a point x** if $\text{tr}_W\{II_{\partial\Omega}\} \geq 0$ for all tangential p -planes $W \subset T_x(\partial\Omega)$ at x .

Theorem 4.7 is the compilation of the following two results.

THEOREM 4.9. *If the domain Ω is locally p -convex, then its boundary $\partial\Omega$ is p -convex.*

THEOREM 4.10. *If the boundary $\partial\Omega$ is p -convex, then the domain Ω is p -convex.*

Before proving these two theorems we make some remarks on boundary convexity.

Remark 4.11. (Local defining functions). Suppose ρ is a smooth function on a neighborhood B of a point $x \in \partial\Omega$ with $\partial\Omega \cap B = \{\rho = 0\}$ and $\Omega \cap B = \{\rho < 0\}$. If $d\rho$ is non-zero on $\partial\Omega \cap B$, then ρ is called a **local defining function for $\partial\Omega$** . It has the property that

$$D_x^2\rho = |\nabla\rho(x)|II_x \quad (4.2)$$

on $\partial\Omega \cap B$. To see this, suppose that e is a vector field tangent to $\partial\Omega$ along $\partial\Omega$, and note that $II(e, e) = \langle n, \nabla_e e \rangle = -\frac{1}{|\nabla\rho|} \langle \nabla\rho, \nabla_e e \rangle$ and $-\langle \nabla\rho, \nabla_e e \rangle = -(\nabla_e e)(\rho) = e(e\rho) - (\nabla_e e)(\rho) = (D^2\rho)(e, e)$. As a consequence we have that $\partial\Omega$ is p -convex at a point x if and only if

$$\text{tr}_W D_x^2\rho \geq 0 \quad \text{for all } p\text{-planes } W \text{ tangent to } \partial\Omega \text{ at } x \quad (4.3)$$

where ρ is a local defining function for $\partial\Omega$. Moreover, (4.3) is independent of the choice of local defining function.

Remark 4.12. (Principal curvatures). Let $\kappa_1 \leq \dots \leq \kappa_{n-1}$ denote the ordered eigenvalues of II_x . Then we have that

$$\partial\Omega \text{ is } p\text{-convex at } x \iff \kappa_1 + \dots + \kappa_p \geq 0. \quad (4.4)$$

Proof. Apply Corollary 2.6 to $A \equiv II$ with $V \equiv T_x\partial\Omega$. ■

We now give the proof of Theorem 4.9, that local p -convexity implies boundary p -convexity.

Lemma 4.13. *If $\partial\Omega$ is not p -convex at a point $x \in \partial\Omega$, then there exists an embedded minimal p -dimensional submanifold M through the point x with*

$$M - \{x\} \subset \Omega \quad \text{in a neighborhood of } x. \quad (4.4)$$

Proof of Theorem 4.9. Assume that $\partial\Omega$ is not p -convex at a point $x \in \partial\Omega$. Let B denote the ϵ -ball about x . It suffices to show that $\Omega \cap B$ is not p -convex. This is done by constructing a “tin can” inside B using Lemma 4.13. We can assume that M is a compact manifold with boundary and $M \subset B$.

Let $M_t \equiv M + t\nu$ denote the translate of M by $t\nu$ where ν is the outward-pointing unit normal to $\partial\Omega$ at x . Choose $r > 0$ sufficiently small that each $M_t \subset \Omega$ for $-r \leq t < 0$. Let K denote the “empty tin can” consisting of the “bottom” M_{-r} and the “label” $\bigcup_{-r \leq t \leq 0} \partial M_t$. Then K is a compact subset of $\Omega \cap B$. Let \hat{K} be its p -convex hull in $\Omega \cap B$.

Since $\partial M_t \subset K$, Proposition 4.2 implies that each $M_t \subset \hat{K}$ for $-r \leq t < 0$. Since \hat{K} is closed in $\Omega \cap B$, this proves that x must be in the \mathbf{R}^n -closure of \hat{K} , i.e., \hat{K} is not compact. Hence, $\Omega \cap B$ is not p -convex. ■

Proof of Lemma 4.13. Suppose $\partial\Omega$ is not p -convex at x . Then there is a tangent p -plane W to $\partial\Omega$ at x with

$$\text{tr}_W\{II_{\partial\Omega}\} < 0. \quad (4.6)$$

We may assume that W is the plane spanned by eigenvectors of II with the smallest eigenvalues. We can then choose euclidean coordinates (t_1, \dots, t_n) with respect to an orthonormal basis e_1, \dots, e_n so that:

- (i) x corresponds to the origin 0,
- (ii) $n = e_n$ is the outward pointing normal to Ω at x .
- (iii) e_1, \dots, e_{n-1} are the eigenvectors of II at x with eigenvalues $\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_{n-1}$
- (iv) $W = \text{span}\{e_1, \dots, e_p\}$

In a neighborhood of 0 our domain can be written as

$$\Omega = \{t_n < f(t_1, \dots, t_{n-1})\}.$$

In particular, $\rho(t) \equiv t_n - f(t_1, \dots, t_{n-1})$ is a local defining function for $\partial\Omega$ near $0 \in \partial\Omega$. By Remark 4.11, since $(\nabla\rho)(0) = e_n$ is a unit vector,

$$D_0^2\rho = -D_0^2f = II_0. \quad (4.7)$$

Hence f has Taylor expansion

$$f(t) = -\frac{1}{2}(\kappa_1 t_1^2 + \dots + \kappa_{n-1} t_{n-1}^2) + O(|t|^3). \quad (4.8)$$

By setting $c \equiv -\frac{1}{p}(\kappa_1 + \dots + \kappa_p)$ we obtain a diagonal matrix $\text{diag}(\kappa_1 + c, \dots, \kappa_p + c)$ with trace zero. The hypothesis (4.6) is equivalent to $c > 0$.

We now restrict attention to the linear subspace $P \equiv \text{span}\{e_1, \dots, e_p, e_n\} = W \oplus \mathbf{R}e_n$, and consider graphs $\{t_n = g(t_1, \dots, t_p)\}$ which are minimal hypersurfaces in P (and therefore in \mathbf{R}^n). We apply the following basic lemma, whose proof is left as an exercise.

Lemma 4.14. *Given $A \in \text{Sym}^2(\mathbf{R}^p)$ with $\text{tr}A = 0$, there exists a real analytic function g defined near the origin with $g(0) = 0$, $(\nabla g)(0) = 0$ and $D_0^2g = A$ such that g satisfies the minimal surface equation.*

We can apply this lemma with $A = -\text{diag}(\kappa_1 + c, \dots, \kappa_p + c)$ obtaining a minimal surface $M = \{(t, g(t)) \in P = \mathbf{R}^{p+1} : |t| < \eta\} \subset \mathbf{R}^n$. The hypothesis $c > 0$ implies that $g(t) < f(t_1, \dots, t_p, 0, \dots, 0)$ if $0 < |t| < \eta$ small. This implies that $M - \{0\} \subset \Omega$, completing the proof of Lemma 4.13 and Theorem 4.9 as well. ■

Now we commence with the proof of Theorem 4.10. Let $\delta(x)$ denote the distance from a point $x \in \Omega$ to the boundary $\partial\Omega$. By the ϵ -collar of $\partial\Omega$ we shall mean the set $\{x \in \Omega : 0 < \delta(x) < \epsilon\}$. Theorem 4.10 is immediate from the next result.

Proposition 4.15.

- (1) *If $\partial\Omega$ is p -convex on a neighborhood of $x_0 \in \partial\Omega$, then $-\log\delta(x)$ is p -plurisubharmonic on the intersection of a neighborhood of x_0 in \mathbf{R}^n with an ϵ -collar of $\partial\Omega$.*
- (2) *If $-\log\delta(x)$ is p -plurisubharmonic on an ϵ -collar of $\partial\Omega$, then Ω is p -convex.*

Summary 4.16. From this proposition and Theorems 4.9 and 4.10 we conclude that

$$\begin{aligned} \Omega \text{ is locally } p\text{-convex} & \iff \partial\Omega \text{ is } p\text{-convex} \\ & \iff -\log\delta(x) \text{ is } p\text{-plurisubharmonic} \\ & \iff \Omega \text{ is } p\text{-convex} \end{aligned} \quad (4.9)$$

Proof of (1). Let II denote the second fundamental form of the hypersurfaces $\{\delta = \epsilon\}$ for $\epsilon \geq 0$, and let $n = \nabla\delta$ denote the inward-pointing normal. An arbitrary p -plane V at a point can be put in a canonical form with basis

$$(\cos\theta)n + (\sin\theta)e_1, e_2, \dots, e_p$$

where n, e_1, \dots, e_p are orthonormal. Set $W \equiv \text{span}\{e_1, \dots, e_p\}$, the *tangential part* of V .

Lemma 4.17.

$$\mathrm{tr}_V \mathrm{Hess}(-\log \delta) = \frac{1}{\delta} \sin^2 \theta \mathrm{tr}_W(II) + \frac{1}{\delta^2} \cos^2 \theta$$

Proof. See Remark after Proposition 5.13 in [HL₁]. ■

Note. This formula holds on any riemannian manifold.

If II has eigenvalues $\kappa_1, \dots, \kappa_{n-1}$ at a point $x \in \partial\Omega$, then let $\kappa_1(\delta), \dots, \kappa_{n-1}(\delta)$ denote the eigenvalues of II at the point a distance δ from x along the normal line. A proof of the following can be found in [GT, §14.6].

Lemma 4.18. *For small $\delta \geq 0$ one has*

$$\kappa_j(\delta) = \frac{\kappa_j}{1 - \delta \kappa_j}, \quad j = 1, \dots, n-1.$$

Corollary 4.19. *Each $\kappa_j(\delta)$ is strictly increasing if $\kappa_j \neq 0$ and $\equiv 0$ if $\kappa_j = 0$.*

We now combine Lemma 4.17 with Corollary 4.19 to conclude that $-\log \delta$ is p -plurisubharmonic.

Remark 4.20. Note that each $\partial\Omega_\epsilon$, where $\Omega_\epsilon \equiv \{\delta > \epsilon\}$, is strictly p -convex and $-\log \delta$ is strictly p -plurisubharmonic if and only if $\partial\Omega$ has no p -flat points, i.e., points where the nullity of $II_{\partial\Omega}$ is $\geq p$.

Proof of (2). By Theorem 4.4 it suffices to prove the existence of a continuous exhaustion function $u : \Omega \rightarrow \mathbf{R}^+$ which is smooth and p -plurisubharmonic outside a compact set in Ω . Such a function is given by setting $u(x) = \max\{-\log \delta(x), -\log(\epsilon/2)\}$. ■

5. Strict Convexity.

Let X be a riemannian manifold.

Definition 5.1. We say that a function $u \in \mathrm{PSH}_p^\infty(X)$ is **strictly p -plurisubharmonic at a point** $x \in X$ if $\mathrm{Hess}_x u \in \mathrm{Int} \mathcal{P}_p(T_x X)$, i.e., if one of the following equivalent conditions holds.

- (1) $\mathrm{tr}_W \mathrm{Hess}_x u > 0$ for all $W \in G(p, T_x X)$.
- (2) $\lambda_1(\mathrm{Hess}_x u) + \dots + \lambda_p(\mathrm{Hess}_x u) > 0$.
- (3) $D_{\mathrm{Hess}_x u} > 0$.

Definition 5.2. The manifold X is called **strictly p -convex** if it admits a proper exhaustion function $u : X \rightarrow \mathbf{R}$ which is strictly p -plurisubharmonic at every point, and it is called **strictly p -convex at infinity** if it admits a proper exhaustion function $u : X \rightarrow \mathbf{R}$ which is strictly p -plurisubharmonic at infinity.

Definition 5.3. The **p -core of X** is defined to be the subset

$$\mathrm{Core}_p(X) \equiv \{x \in X : u \text{ is not strict at } x \text{ for all } u \in \mathrm{PSH}_p^\infty(X)\}$$

THEOREM 5.4. *Suppose X is a riemannian manifold.*

(1) X admits a smooth strictly p -plurisubharmonic function \iff

$$\text{Core}(X) = \emptyset.$$

(2) X is strictly p -convex, i.e., X admits a smooth strictly p -plurisubharmonic proper exhaustion function \iff

$$\text{Core}(X) = \emptyset \text{ and } X \text{ is } p\text{-convex.}$$

(3) X is strictly p -convex at infinity \iff

$$\text{Core}(X) \text{ is compact and } X \text{ is } p\text{-convex.}$$

Proof. Part (1) is a special case of Theorem 4.2 in [HL₈]; Part (2) is a special case of 4.8 in [HL₈]; and Part (3) is a special case of Theorem 4.11 in [HL₈]. ■

Proposition 5.5. *Every compact p -dimensional minimal submanifold M without boundary in X is contained in $\text{Core}_p(X)$. If instead the boundary $\partial M \neq \emptyset$ and M is connected, then*

$$M \subset \widehat{\partial M}.$$

Proof. For the first assertion, apply Theorem 2.10 and the maximum principle to conclude the restriction of any smooth p -plurisubharmonic function to M is constant. The second assertion is Proposition 4.2. ■

This provides an analogue of the support Lemma 3.2 in [HL₂].

Corollary 5.6. *Suppose $M \subset X$ is a compact p -dimensional minimal submanifold with possible boundary. Then*

$$M \subset \widehat{\partial M} \cup \text{Core}(X).$$

The following is a well known result of Sha [S₁] and Wu [Wu]. It is also an easy special case of Theorem 6.2 in [HL₁] (see also Theorem 4.16 in [HL₈]).

Proposition 5.7. *If X is strictly p -convex, then X has the homotopy-type of a complex of dimension $\leq p - 1$.*

Proof. Let $f : X \rightarrow \mathbf{R}$ be a strictly p -plurisubharmonic exhaustion function, and by approximation assume that f has non-degenerate critical points. We claim that at any critical point $x \in X$ the index of $\text{Hess}f$ is $\leq p - 1$. If not, there must be a subspace $V \subset T_x X$ of dimension $\geq p$ on which $\text{Hess}f$ is < 0 , contrary to assumption. ■

Proposition 5.8. *Let $\Omega \subset X$ be a domain with a smooth boundary $\partial\Omega$ which is strictly p -convex, i.e. $\text{tr}_W \{II_{\partial\Omega}\} > 0$ for all p -planes tangent to $\partial\Omega$. Then Ω is strictly p -convex at infinity. Furthermore, if $\text{Core}(X) = \emptyset$, then Ω is strictly p -convex.*

Proof. The strict p -convexity of the boundary implies that the function $-\log\delta(x)$ is strictly p -plurisubharmonic on a collar of $\partial\Omega$ in Ω using Lemma 4.17 (which holds on riemannian manifolds). Now apply Lemma 4.5. The second statement follows from Theorem 5.4 (2). ■

The case of a domain $\Omega \subset X$ with a (not necessarily strict) p -convex boundary is more delicate. In the euclidean case treated in Section 4 we required an exact calculation of the second fundamental form of the equidistant sets to the boundary (Lemma 4.18). In the general riemannian case this calculation involves Jacobi fields along geodesics emanating normally from the boundary. This requires a condition on the Riemann curvature tensor.

To each unit tangent vector $e \in T_x X$ there is a quadratic form $K_e \in \text{Sym}^2(T_x X)$ given by $K_e(v) = \|v\|^2 K(e \wedge v)$ where $K(e \wedge v)$ denotes the sectional curvature of the 2-plane $\text{span}\{e, v\}$. The curvature of X is said to be **p -positive** (or **strictly p -positive**) if K_e is p -positive (or strictly p -positive) for all e .

THEOREM 5.9. (H. Wu [Wu]). *Let $\Omega \subset X$ be a domain with a smooth p -convex boundary $\partial\Omega$. If the curvature of X is strictly p -positive in a neighborhood of $\partial\Omega$, then Ω is strictly p -convex at infinity. Moreover, if the curvature is p -positive on Ω and either*

- (a) *curvature of X is strictly p -positive on a neighborhood of $\partial\Omega$, or*
- (b) *the boundary $\partial\Omega$ is strictly p -convex,*

then Ω is strictly p -convex.

Related results have been proved by J-Ping Sha [S₁].

It is natural to raise the existence question for p -convex domains. The next three results give particularly nice answers.

Proposition 5.10. *Let $N \subset X$ be any compact submanifold without boundary of dimension $\leq p - 1$. Then $f(x) \equiv \text{dist}(x, N)^2$ is strictly p -plurisubharmonic in a neighborhood of N .*

Proof. See the proof of Theorem 6.4 in [HL₁]. ■

THEOREM 5.11. *Let $N \subset X$ be any (not necessarily compact) submanifold of dimension $\leq p - 1$. Then N has a fundamental neighborhood system of strictly p -convex domains.*

Proof. See the proof of Theorem 6.4 in [HL₁] (see also Theorem 4.17 in [HL₈]). ■

THEOREM 5.12. (J.-P. Sha [S₂]). *Let $\Omega \subset\subset X$ be a domain with a smooth, strictly p -convex boundary. Suppose $\Omega' \subset\subset X$ is a domain with smooth boundary constructed from Ω by attaching a handle of dimension $\leq p - 1$ along the exterior side of Ω . Then Ω' is ambiently isotopic to a domain with strictly p -convex boundary.*

6. Minimal Varieties and their Associated Hulls.

In this section we introduce the *minimal current hull* of a compact set K in a riemannian manifold X , and relate it to the p -convex hull \hat{K} discussed in §4. This hull will be defined using the group $\mathcal{R}_p(X)$ of p -dimensional rectifiable currents with compact support in X (cf. [F], [Si], [M], etc.). These creatures enjoy many nice properties. They can be usefully considered as compact oriented p -dimensional manifolds with singularities and integer multiplicities, and readers unfamiliar with the general theory can think of them simply as submanifolds.

Of importance here is the following general structure theorem. Associated to each $T \in \mathcal{R}_p(X)$ is a Radon measure $\|T\|$ on X and a $\|T\|$ -measurable field of unit p -vectors \vec{T} such that for any smooth p -form ω on X ,

$$T(\omega) = \int_X \omega(\vec{T}) d\|T\|. \quad (6.1)$$

(Recall [deR] that the p -currents are the topological dual space to the space of smooth p -forms.) In particular, every $T \in \mathcal{R}_p(X)$ has a finite mass

$$\mathbf{M}(T) = \int_X d\|T\|.$$

Example. When T corresponds to integration over a compact oriented submanifold with boundary, of finite volume $M \subset X$, one has $\|T\| = \mathcal{H}_p|_M$ (\mathcal{H}_p = Hausdorff measure), \vec{T}_x corresponds to the oriented tangent plane $T_x M$, and $\mathbf{M}(T) = \mathcal{H}_p(M)$ = the riemannian volume of M .

Definition 6.1. A current $T \in \mathcal{R}_p(X)$ is called **minimal** or **stationary** if for all smooth vector fields V on X which vanish on a neighborhood of the support of ∂T , one has

$$\left. \frac{d}{dt} \mathbf{M}((\varphi_t)_* T) \right|_{t=0} = 0, \quad (6.2)$$

where φ_t denotes the flow generated by V on a neighborhood of the support of T .

Each smooth vector field on X defines a smooth bundle map $\mathcal{A}^V : TX \rightarrow TX$ given on a tangent vector W by

$$\mathcal{A}^V(W) \stackrel{\text{def}}{=} \nabla_W V. \quad (6.3)$$

This determines the derivation $D_{\mathcal{A}^V} : \Lambda^p TX \rightarrow \Lambda^p TX$ as in Section 2. Proof of the following can be found in [LS] or [L].

THEOREM 6.2. (The First Variational Formula). Fix $T \in \mathcal{R}_p(X)$ and let V, φ_t be as above. Then

$$\left. \frac{d}{dt} \mathbf{M}((\varphi_t)_* T) \right|_{t=0} = \int_X \langle D_{\mathcal{A}^V} \vec{T}, \vec{T} \rangle d\|T\| = \int_X \text{tr}_{\vec{T}} (\mathcal{A}^V) d\|T\| \quad (6.4)$$

Suppose now that $V = \nabla u$ for a smooth function u on X . Then

$$\mathcal{A}^{\nabla u} = \text{Hess } u, \quad (6.5)$$

considered as an endomorphism of TX . To see this note that $\langle \mathcal{A}^{\nabla u}(W), U \rangle \langle \nabla_W(\nabla u), U \rangle = W \langle \nabla u, U \rangle - \langle \nabla u, \nabla_W U \rangle = (WU - \nabla_W U)u = (\text{Hess } u)(W, U)$. Hence, we have the following.

THEOREM 6.4. *If $V = \nabla u$, then*

$$\left. \frac{d}{dt} \mathbf{M}((\varphi_t)_* T) \right|_{t=0} = \int_X \operatorname{tr}_{\overrightarrow{T}} (\operatorname{Hess} u) d\|T\| \quad (6.6)$$

Corollary 6.5. *Suppose $T \in \mathcal{R}_p(X)$ is a minimal current, and let u be any smooth p -plurisubharmonic function which vanishes on a neighborhood of $\operatorname{supp}(\partial T)$. Then*

$$\operatorname{tr}_{\overrightarrow{T}} (\operatorname{Hess} u) \equiv 0 \quad \text{on } \operatorname{supp}(T).$$

If $T = [M]$ is the current associated to a connected p -dimension minimal submanifold, and if the p -plurisubharmonic function u and its gradient both vanish at points of ∂M , then

$$u|_M \equiv 0 \quad \text{or, if } \partial M = \emptyset, u|_M \equiv \text{constant}.$$

Proof. The first statement follows directly from (6.2), (6.6) and the fact that $\operatorname{tr}_W \operatorname{Hess} u \geq 0$ on all tangent p planes W . Now when $T = [M]$ for a minimal submanifold M , we have $\operatorname{tr}_{T_x M} (\operatorname{Hess}_x u) = \Delta_M(u|_M)$ where Δ_M is the Laplace-Beltrami operator of M in the induced metric (see Proposition 2.10 in [HL₁]). Hence, $u|_M$ is harmonic on M with constant boundary values (if $\partial M \neq \emptyset$). The conclusion follows from the maximum principle. ■

Corollary 6.6. *Let $K \subset X$ be a compact subset and suppose $T \in \mathcal{R}_p(X)$ is a minimal current such that $\operatorname{supp}(\partial T) \subset K$. Then*

$$\operatorname{supp}(T) \subset \hat{K} \cup \operatorname{Core}(X).$$

Proof. Suppose $x \notin \hat{K}$. Then by the p -plurisubharmonic analogue of Lemma 4.2 in [HL₁] there exists a smooth non-negative p -plurisubharmonic function u , which is zero on a neighborhood of K and satisfies $u(x) > 0$, and furthermore, if $x \notin \operatorname{Core}(X)$, then u can be chosen to be strict at x . This violates (6.6). ■

Definition 6.7. Given a compact subset $K \subset X$, we define the **minimal p -current hull** to be the set

$$\hat{K}_{\min} = \bigcup \operatorname{supp}(T)$$

where the union is taken over all minimal $T \in \mathcal{R}_p(X)$ with $\operatorname{supp}(\partial T) \subset K$.

Note that \hat{K}_{\min} contains all compact minimal oriented p -dimensional submanifolds with boundary in K .

Corollary 6.7.

$$\hat{K}_{\min} \subset \hat{K} \cup \operatorname{Core}(X).$$

If X supports a global strictly p -plurisubharmonic function, then $\operatorname{Core}(X) = \emptyset$. For example, $|x|^2$ is such a function on \mathbf{R}^n .

Question 6.8. Suppose $\Gamma \subset \mathbf{R}^n$ is a compact $(p-1)$ -dimensional submanifold which bounds exactly one minimal p -current in \mathbf{R}^n and that current is an oriented submanifold M . How close does $\hat{\Gamma}$ come to approximating M ?

General Minimal Currents

Much of what is said above carries over to general compactly supported currents of finite mass. These are exactly the currents which can be represented as in (6.1) with the provision that the $\|T\|$ -measurable field \vec{T} of unit p -vectors is no longer required to be simple $\|T\|$ -a.e.. Definition 6.1 makes sense for such currents, and the first variational formula

$$\left. \frac{d}{dt} \mathbf{M}((\varphi_t)_* T) \right|_{t=0} = \int_X \langle D_{\mathcal{A}^V} \vec{T}, \vec{T} \rangle d\|T\|$$

holds. If $V = \nabla u$ where $u \in \text{PSH}_p^\infty(X)$, then by Definition 2.1 (3) we know that $D_{\mathcal{A}^V} \geq 0$. Furthermore, at any point where u is strict, we have $D_{\mathcal{A}^V} > 0$. The arguments for Corollaries 6.5 and 6.6 give the following.

Proposition 6.9. *Let T be a minimal p -dimensional current of finite mass and compact support on X , and let u be any smooth p -plurisubharmonic function which vanishes on a neighborhood of $\text{supp}(\partial T)$. Then*

$$\langle D_{\text{Hess } u} \vec{T}, \vec{T} \rangle = 0 \quad \|T\| - a.e.$$

Furthermore,

$$\text{supp}(T) \subset \widehat{\partial T} \cup \text{Core}(X).$$

Thus the minimal current hull \widehat{K}_{\min} can be expanded to contain the supports of all minimal currents with boundary supported in K , and Corollary 6.7 remains true.

Examples. Minimal non-rectifiable currents abound in geometry. Any positive (p, p) -current on a Kähler manifold X is minimal. This observation extends to positive ϕ -currents on any calibrated manifold (X, ϕ) (see [HL₀, §?]). Any foliation current whose leaves are minimal p -submanifolds is a minimal current.

There are two basic cases of smooth minimal currents which are interesting. Let T be a smooth d -closed current of dimension $n-1$ (degree 1). Then T is simply a closed 1-form and can be written locally as $T = df$ for a smooth function f . In a neighborhood of any point where $df \neq 0$, the minimality condition is equivalent to the 1-Laplace Equation:

$$d \left(* \frac{df}{\|df\|} \right) = 0.$$

which says that $* \frac{df}{\|df\|}$ calibrates the level hypersurfaces of f . In particular, the level sets of f are minimal varieties.

Let T be a smooth d -closed current of dimension 1. Then T can be expressed on a compactly supported 1-form α as $T(\alpha) = \int_X \alpha(V) d\text{vol}_X$ where V is a smooth vector field. Minimality is the condition that

$$\nabla_V \left(\frac{V}{\|V\|} \right) = 0,$$

which means exactly that the (reparameterized) flow lines of V are geodesics in X , and the d -closed condition is equivalent to

$$\text{div}(V) = 0.$$

7. Potential Theory.

The concept of p -plurisubharmonicity can be carried over to upper semi-continuous functions, and these more general objects have proved to be quite useful. This has been studied in [HL₄], [HL₃], [HL₅] and [HL₇]. We give a brief summary of the known results here.

Let X be a riemannian manifold and denote by $\text{USC}(X)$ the space of upper semi-continuous $[-\infty, \infty)$ -valued functions on X . By a **test function** for $u \in \text{USC}(X)$ at a point x we mean a C^2 -function φ , defined near x , such that $u \leq \varphi$ near x and $u(x) = \varphi(x)$.

Definition 7.1. A function $u \in \text{USC}(X)$ is **p -plurisubharmonic** if for each $x \in X$ and each test function φ for u at x , the riemannian hessian $\text{Hess}_x \varphi$ at x is p -positive. The space of these functions is denote $\text{PSH}_p(X)$.

The following basic facts can be found for example in [HL₅, Prop. 2.6].

THEOREM 7.2.

- (a) (Maximum Property) If $u, v \in \text{PSH}_p(X)$, then $w = \max\{u, v\} \in \text{PSH}_p(X)$.
- (b) (Coherence Property) If $u \in \text{PSH}_p(X)$ is twice differentiable at $x \in X$, then $\text{Hess}_x u$ is p -positive.
- (c) (Decreasing Sequence Property) If $\{u_j\}$ is a decreasing ($u_j \geq u_{j+1}$) sequence of functions with all $u_j \in \text{PSH}_p(X)$, then the limit $u = \lim_{j \rightarrow \infty} u_j \in \text{PSH}_p(X)$.
- (d) (Uniform Limit Property) Suppose $\{u_j\} \subset \text{PSH}_p(X)$ is a sequence which converges to u uniformly on compact subsets to X , then $u \in \text{PSH}_p(X)$.
- (e) (Families Locally Bounded Above) Suppose $\mathcal{F} \subset \text{PSH}_p(X)$ is a family of functions which are locally uniformly bounded above. Then the upper semicontinuous regularization v^* of the upper envelope

$$v(x) = \sup_{u \in \mathcal{F}} u(x)$$

belongs to $\text{PSH}_p(X)$.

There are inclusions $\text{PSH}_1(X) \subset \text{PSH}_2(X) \subset \cdots \subset \text{PSH}_n(X)$, and $\text{PSH}_n(X)$ is the set of classical subharmonic functions on X . In particular, they satisfy the maximum principle.

If $u \in \text{USC}(X)$ has the property that its (local) restrictions to p -dimensional minimal submanifolds are subharmonic, then $u \in \text{PSH}_p(X)$. This follows from the fact that given any $x \in X$ and $W \in G(p, T_x X)$ there exists a minimal submanifold M with $T_x M = W$. A basic non-trivial result is the converse. It is a special case of Theorem 6.2 in [HL₇].

THEOREM 7.3. (The Restriction Theorem). *The restriction of any $u \in \text{PSH}_p(X)$ to a p -dimensional minimal submanifold $M \subset X$ is subharmonic (or $\equiv -\infty$) on M with respect to the induced riemannian structure.*

8. Viscosity Solutions of the Homogeneous p -Monge-Ampère Equation.

Consider any closed subset $F \subset \text{Sym}^2(\mathbf{R}^n)$. We say that an upper semi-continuous function u on a domain $X \subset \mathbf{R}^n$ is F -subharmonic if for all $x \in X$ and all test functions φ for u at x , we have $\text{Hess}_x \varphi \in F$. Somewhat surprisingly, the set $F(X)$ of these functions enjoys all the properties of Theorem 7.2 (see [HL₅, Prop. 2.6]). However, in order for this definition to be meaningful, F must satisfy the *Positivity Condition*:

$$F + \mathcal{P}_1(\mathbf{R}^n) \subset F. \quad (8.1)$$

Then for $u \in C^2(X)$, u is F -subharmonic if and only if $\text{Hess}_x u \in F$ for all $x \in X$.

When F is invariant under the natural action of the orthogonal group, this definition carries over to any riemannian manifold (cf. [HL₅]).

Associated to F is the *dual subequation*

$$\tilde{F} = \sim (-\text{Int} F) = -(\sim \text{Int} F)$$

which satisfies the Positivity Condition (8.1) if F does. Note that $\partial F = F \cap (-\tilde{F})$.

Definition 8.1. A (real-valued) function u on X is called **F -harmonic** if

$$u \in F(X) \quad \text{and} \quad -u \in \tilde{F}(X).$$

A C^2 -function is F -harmonic if and only if $\text{Hess}_x u \in \partial F$ for all $x \in X$.

The case of interest in this paper is $F = \mathcal{P}_p^{(k)}$.

Definition 8.2. A function u on X is a **solution of the k^{th} branch of the homogeneous p -Monge-Ampère equation** if u is $\mathcal{P}_p^{(k)}$ -harmonic.

If u is C^2 , then it is a solution if and only if $\text{Hess}_x u \in \partial \mathcal{P}_p^{(k)}$ for all $x \in X$. As a result of (3.4) this implies that

$$\text{MA}_p(u) = 0 \quad \text{on } X$$

Since the sets $\mathcal{P}_p^{(k)}$ are orthogonally invariant, these definitions and remarks make sense on any riemannian manifold X using the riemannian hessian.

Embedded in Definition 8.2 is the requirement that $-u$ satisfies the dual subequation $\tilde{\mathcal{P}}_p^{(k)}$. The dual of the subequation \mathcal{P}_p is the set

$$\begin{aligned}\tilde{\mathcal{P}}_p &= \{A \in \text{Sym}^2(V) : \exists W \in G(p, V) \text{ with } \text{tr}_W A \geq 0\} \\ &= \{A \in \text{Sym}^2(V) : \lambda_{n-p+1}(A) + \cdots \lambda_n(A) \geq 0\} \\ &= \{A \in \text{Sym}^2(V) : \lambda_{\max}(D_A) \geq 0\}.\end{aligned}$$

These calculations follow from the three ways of describing $\mathcal{P}_p(V)$ given in Definition 2.1. More generally, using the terminology of Section 3, the dual of $\mathcal{P}_p^{(k)} = \{A : \lambda_{I_k}(D_A) \geq 0\}$ is

$$\tilde{\mathcal{P}}_p^{(k)} = \mathcal{P}_p^{(k^*)}$$

where $k^* = \binom{n}{p} - k + 1$.

9. The Dirichlet Problem for the Homogeneous p -Monge-Ampère Equation.

Proposition 5.9 asserts the existence of many strictly p -convex domains. It turns out that on such domains one can uniquely solve the Dirichlet problem for the homogeneous p -Monge-Ampère Equation, in fact for every branch of that equation.

THEOREM 9.1. *Let $\Omega \subset\subset X$ be a domain with a smooth strictly p -convex boundary $\partial\Omega$ and with $\text{Core}(\Omega) = \emptyset$ in an n -dimensional riemannian manifold X . Fix an integer k with $1 \leq k \leq \binom{n}{p}$. Then for each continuous function $\varphi \in \mathbf{C}(\partial\Omega)$, there exists a unique continuous function $u \in C(\overline{\Omega})$ such that:*

(a) u is a solution of the k^{th} branch of the homogeneous p -Monge-Ampère equation on Ω , and

(b) $u|_{\partial\Omega} = \varphi$.

Note 9.2. The hypothesis on Ω is equivalent to the requirement that there exists a smooth strictly p -plurisubharmonic function ρ , defined on a neighborhood of $\overline{\Omega}$, with $\Omega = \{\rho < 0\}$ and $\nabla\rho \neq 0$ on $\partial\Omega$.

Note 9.3. This theorem remains true if one replaces the operator $\text{MA}_p(u) = \det\{D_{\text{Hess } u}\}$ with the operator $\sigma_\ell\{D_{\text{Hess } u}\}$ where σ_ℓ denotes the ℓ^{th} elementary function, $1 \leq \ell \leq \binom{n}{p}$. In this case the cone $\mathcal{P}_p(V)$ is replaced by the convex cone

$$\mathcal{S}_{p,\ell}(V) \equiv \{A : \sigma_1(D_A) \geq 0, \sigma_2(D_A) \geq 0, \dots, \sigma_\ell(D_A) \geq 0\}$$

Again there are branches, and the Dirichlet problem can be solved for each of these branches. An important point is that $\mathcal{P}_p(V)$ is a monotonicity cone for each of these equations, i.e., $\mathcal{S}_{p,\ell}(V) + \mathcal{P}_p(V) \subset \mathcal{S}_{p,\ell}(V)$.

Note 9.4. For all equations except the main branch of $\text{MA}_p(u)$, Theorem 9.1 remains true under weaker convexity assumptions on the boundary $\partial\Omega$ (see [HL₅]).

All of the above represents a special case of results established in [HL₅]. A useful elaboration of the algebra involved is given in [HL₆]

Note 9.5. Certain versions of the inhomogeneous equation (in the case $k = 1$) can also be solved using the notion of affine jet equivalence. The reader is referred to [HL₅] for details.

Appendix A: Extreme Rays in the Convex Cone $\mathcal{P}_p(V)$.

For each unit vector $e \in V$, orthogonal projection onto the line $\text{span}(e)$ can be written as the symmetric product $e \circ e$. Recall the classical fact that the extreme rays in $\mathcal{P}_1(V) = \{A : A \geq 0\}$ are exactly those generated by $e \circ e$ for $e \in V$.

Proposition A.1. *The extreme rays in $\mathcal{P}_p(V)$ are of two types if $1 < p < n - 1$.*

$$(1) \quad \frac{1}{p}I - e \circ e, \text{ or}$$

$$(2) \quad e \circ e$$

where e is a unit vector in V .

If $p = 1$, only case (2) occurs.

If $p = n - 1$, only case (1) occurs.

By using the natural inner product on $\text{Sym}^2(V)$, given by $\langle A, B \rangle \equiv \text{tr}AB$, and identifying $G(p, V)$ with a subset of $\text{Sym}^2(V)$ via the correspondence $W \cong P_W$ (orthogonal projection onto the subspace W), Definition 2.1 (1) can be restated as:

$$\mathcal{P}_p(V) = \{A \in \text{Sym}^2(V) : \langle A, P_W \rangle \geq 0 \ \forall W \in G(p, V)\}. \quad (1)'$$

Let $\mathcal{P}_+ \subset \text{Sym}^2(V)$ denote the convex cone (with vertex at the origin) on the subset $G(p, V) \subset \text{Sym}^2(V)$. Now (1)' is the statement that the polar cone of \mathcal{P}_+ is

$$\mathcal{P}^+ \equiv (\mathcal{P}_+)^0 = \mathcal{P}_p(V).$$

It is easy to see that the extreme rays in \mathcal{P}_+ (the polar of \mathcal{P}^+) are given by the orthogonal projections P_W with $W \in G(p, V)$.

Proof of Proposition A.1. Given an orthonormal basis e_1, \dots, e_n for V , the axis p -planes can be written as $E_I = \text{span}\{e_{i_1}, \dots, e_{i_p}\}$ with $i_1 < \dots < i_p$ increasing. Let $P_I \equiv P_{E_I}$ denote orthogonal projection onto the axis p -plane E_I . Now $P_I = \text{diag}(\epsilon_1, \dots, \epsilon_n) \in \mathcal{D} = \mathbf{R}^n$ where $\epsilon_i = 1$ if $i \in I$ and $\epsilon_i = 0$ if $i \notin I$.

Under the action of O_n on $\text{Sym}^2(V)$ the set \mathcal{D} of diagonal matrices form an n -dimensional cross-section. Define $\mathbf{P}_+ \equiv \mathcal{P}_+ \cap \mathcal{D}$. This is the convex cone on the set of projections $P_I \equiv P_{E_I}$ to the axis p -planes thus the P_I are the vertices of \mathbf{P}_+ . The polar cone of \mathbf{P}_+ in \mathcal{D} , namely $\mathcal{P}^+ \cap \mathcal{D}$, will be denoted by \mathbf{P}^+ . It suffices to determine the extreme rays of the polygonal cone $\mathbf{P}^+ \subset \mathcal{D} \cong \mathbf{R}^n$.

Fact 1. The vertices (or extreme rays) of \mathbf{P}^+ are the matrices $A = \text{diag}(\lambda_1, \dots, \lambda_n) \in \mathbf{P}^+$ whose face

$$F_A = \text{span}\{P_I : \langle P_I, A \rangle = 0\}$$

is of dimension $n - 1$.

We may assume that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. Let k denote the number of strictly negative eigenvalues of A . Of course, $k < p$.

Fact 2. Assume $A \in \mathbf{P}^+$. If $\langle P_I, A \rangle = 0$, then $\{1, 2, \dots, k\} \subset I$, i.e., E_I contains e_1, \dots, e_k .

Proof. Suppose $e_j \notin E_I$ for some j , $1 \leq j \leq k$. Since $\lambda_{i_1} + \dots + \lambda_{i_p} = 0$ and $k < p$, we have $\lambda_{i_p} \geq 0$. Replace i_p by j , i.e., replace I by $J = (I - \{i_p\}) \cup \{j\}$. This replaces $\lambda_{i_p} \geq 0$ by $\lambda_j < 0$ forcing the new trace $\langle P_J, A \rangle$ to be < 0 .

Lemma A.2. If $A \in \mathbf{P}^+$ is extreme, then A can have at most one strictly negative eigenvalue.

Proof. If A has two negative eigenvalues, then by Fact 2, each E_I with $\langle P_I, A \rangle = 0$ must contain e_1 and e_2 , i.e., $P_I = \text{diag}(1, 1, \dots)$. Hence, $P \equiv \text{diag}(-1, 1, 0, \dots, 0) \perp F_A$. Of course, $A \perp F_A$. Since P and A are linearly independent, F_A must have codimension ≥ 2 . ■

Now suppose that $A \in \mathcal{P}^+$ is one of the vertices of \mathcal{P}^+ , i.e., (A.1) holds.

Positive Case. Suppose all the eigenvalues of A are ≥ 0 . Let $0 = \lambda_1 = \dots = \lambda_\ell < \lambda_{\ell+1} \leq \dots \leq \lambda_n$ define ℓ . For each P_I with $\langle P_I, A \rangle = 0$, E_I must be contained in $\text{span}\{e_1, \dots, e_\ell\}$, the zero-eigenspace of A . In particular $e_j \circ e_j \perp F_A$ for $j = \ell + 1, \dots, n$. Since these e_j are linearly independent, Fact 1 implies that there can be at most one of them, i.e., $\ell = n - 1$. Hence we may assume $A = \text{diag}(0, 0, \dots, 0, 1)$, in which case $F_A = \text{span}\{P_I : n \notin I\}$. If $p = n - 1$, then $\dim F_A = 1$, so this case cannot occur. If $1 \leq p \leq n - 2$, then $\dim F_A = n - 1$, so $A = e_n \circ e_n$ can occur.

Negative Case. By the Lemma and rescaling we may assume that

$$-1 = \lambda_1 < 0 \leq \lambda_2 \leq \dots \leq \lambda_n$$

(This case does not occur if $p = 1$.) Set $\mu_j = \lambda_{j+1}$ for $j = 1, \dots, n - 1$.

Now $\langle P_I, A \rangle = 0$ implies that E_I contains e_1 by Fact 2. Set

$$J \equiv I - \{1\} \quad \text{and} \quad B = \text{diag}(\mu_1, \dots, \mu_{n-1}).$$

Then

$$\langle P_I, A \rangle = 0 \quad \Leftrightarrow \quad -1 + \langle P_J, B \rangle = 0$$

Set

$$F_B \equiv \text{span}\{P_J : \langle P_J, B \rangle = 1 \text{ where } J \text{ is increasing and } |J| = p - 1\}.$$

Now the dimension of F_B is equal to $n - 1$ if and only if A is extreme. This happens when

$$\mu_1 = \mu_2 = \dots = \mu_{n-1} = \frac{1}{p-1},$$

so that $\langle P_J, B \rangle = 1$ for all J . Thus

$$A = -e_1 \circ e_1 + \frac{1}{p-1} \sum_{j=2}^n e_j \circ e_j, \quad \text{or equivalently} \quad \frac{p-1}{p} A = \frac{1}{p} I - e_1 \circ e_1$$

as desired. ■

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